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ON COMPLEX MANIFOLDS POLARIZED BY AN AMPLE LINE BUNDLE OF SECTIONAL GENUS $q(X)+2$

YOSHIAKI FUKUMA
(福間 慶明)

Naruto University of Education
E-mail: fukuma@naruto-u.ac.jp

Let X be a smooth projective variety defined over the complex number field, and let L be a line bundle over X. Then (X, L) is called a polarized (resp. quasi-polarized) manifold if L is ample (resp. nef and big). For such pair (X, L), the delta genus $\Delta(L)$ and the sectional genus $g(L)$ are defined by the following formula:

$$\Delta(L) := n + L^n - h^0(L),$$

$$g(L) := 1 + \frac{1}{2}(K_X + (n-1)L)L^{n-1},$$

where $h^0(L) = \dim H^0(L)$, and K_X is the canonical divisor of X.

In this report, we will state some recent results about the sectional genus of quasi-polarized manifolds, and we will propose some conjectures and problems.

The following results are known for the fundamental properties of the sectional genus;

(1) The value of $g(L)$ is a non-negative integer. (Fujita [Fj1], Ionescu [I])

(2) There exists a classification of polarized manifolds (X, L) with the sectional genus $g(L) \leq 2$. (Fujita [Fj1], [Fj2], Ionescu [I], Beltrametti-Lanteri-Palleschi [BLP], e.t.c.)

(3) Let (X, L) be a polarized manifold with $\dim X = n$. For any fixed n and $g(L)$, there are only finitely many deformation types of polarized masnifolds except scrolls over smooth curves. (For the definition of deformation types of polarized manifolds, see Chapter II, §13 in [Fj4].)

Here we give the definition of a scroll over a variety.

Definition. Let (X, L) be a quasi-polarized manifold with $\dim X = N$, and let Y be a projective variety with $\dim Y = m$ and $N > m \geq 1$. Then (X, L) is called a scroll over Y if there exists a surjective morphism $\pi : X \to Y$ such that any fiber F of π is isomorphic to P^{N-m} and $L_F \cong \mathcal{O}(1)$.

Here we consider (3) above. By (3), if (X, L) is not a scroll over a smooth curve, then the topological invariant of X is expected to be bounded by using some invariant of L. Here we mainly consider the irregularity $q(X) := \dim H^1(\mathcal{O}_X)$ of X. If (X, L) is a scroll over a smooth curve, then we can easily get that $g(L) = q(X)$. So by considering the fact (3) above, Fujita propose the following conjecture;

Conjecture 1. ([Fj3]) *Let (X, L) be a quasi-polarized manifold. Then $g(L) \geq q(X)$.*

For the time being, this conjecture is true if (X, L) is one of the following;

(1) $n = 2$, $\kappa(X) \leq 1$ ([Fk2]),
(2) $n = 2$, $\kappa(X) = 2$, $h^0(L) \geq 1$ ([Fk2]),
(3) $n = 3$, $h^0(L) \geq 2$ ([Fk7]),
(4) $\kappa(X) = 0, 1$, $L^n \geq 2$ ([Fk3]),
(5) $\dim \mathrm{Bs}\,|L| \leq 1$ (For the case in which $\dim \mathrm{Bs}\,|L| \leq 0$, see [Fk6], and for the case in which $\dim \mathrm{Bs}\,|L| = 1$, this result is unpublished).

So our first goal is to prove that this conjecture is true if $n = 2$, $\kappa(X) = 2$, and $h^0(L) = 0$.

Here we give some comments about Conjecture 1. First, for a polarized surface (X,L) with $\kappa(X) \geq 0$ we explain a relation between the value of $g(L) - q(X)$ and the type of divisor $D \in |L|$. Let

(X, L) be a quasi-polarized surface with $\kappa(X) \geq 0$ and $h^0(L) > 0$. Let D be an effective divisor on X which is linearly equivalent to L. Let $D = \sum_i a_i C_i$, where C_i is an irreducible reduced curve and $a_i > 0$ for each i. We take a birational morphism $\mu : X^{\alpha} \to X$ such that $C_1^* \cap C_2^* \cap C_3^* = \phi$ for any distinct three irreducible components C_1^*, C_2^* and C_3^* of $\mu^*(D)$, and if two irreducible curves C_i^* and C_j^* of $\mu^*(D)$ intersect at x, then the intersection number $i(C_i^*, C_j^*; x) = 1$, where $i(C_i^*, C_j^*; x)$ is the intersection number of C_i^* and C_j^* at $x \in C_i^* \cap C_j^*$. Let $\mu_i : X_i \to X_{i-1}$ be one point blowing up such that $\mu = \mu_1 \circ \mu_2 \circ \cdots \circ \mu_t$ and let $D_{\mathrm{red}} = B_0$. Let $(\mu_i^*(B_{i-1}))_{\mathrm{red}} = B_i$ and $B_i = \mu_i^*(B_{i-1}) - b_i E_i$, where E_i is a (-1)-curve such that $\mu_i(E_i) =$ point. Then $b_i \geq 1$. Let $D^{\beta} = \sum_i C_{\beta,i}$ and $\mu^{\gamma} : X^{\gamma} \to X^{\beta}$ be a resolution of singular points $S = \bigcup_i \mathrm{Sing}(C_{\beta,i})$.

Let $\mu_x : X_x^{\beta,i,t_x} \to X_x^{\beta,i,t_x-1} \to \cdots \to X_x^{\beta,i,0}$ be a resolution of singularity at $x \in \mathrm{Sing}(C_{\beta,i})$, where $\mu_x^k : X_x^{\beta,i,k} \to X_x^{\beta,i,k-1}$ is one point blowing up.

Let $(\mu_x^k)^*(D^{\beta,k-1}) = D^{\beta,k} + m(k,x)E^k$, where E^k is (-1)-curve of μ_x^k such that $\mu_x^k(E^k) =$ point and $D^{\beta,k}$ is the strict transform of $D^{\beta,k-1}$ for each k.

By using the above notation, we get the following result;

Theorem 1. ([Fk5]) *Let (X, L) be a polarized surface. Assume that $\kappa(X) \geq 0$ and $h^0(L) > 0$. Let $D \in |L|$ be an effective divisor which is linearly equivalent to L. Then the following inequality holds;*

$$g(D) \geq q(X) + \sum_{x_j \in S} \sum_{k=1}^{t_{x_j}} \frac{m(k,x_j)(m(k,x_j)-1)}{2} + \sum_{i=1}^{t} \frac{b_i(b_i-1)}{2}.$$

Proof. See [Fk5].

By this theorem, if the value of $g(L) - q(X)$ is small, then the singularities of $\mathrm{Supp}\, D$ is simple.

Next we consider the dimension of the global section of the adjoint bundle $K_X+(n-1)L$. The value of $g(L)-q(X)$ is thought to control the value of $h^0(K_X+(n-1)L)$. In [Fk6], the author proposed the following conjecture;

Conjecture 2. ([Fk6]) *Let (X, L) be a quasi-polarized manifold with* $\dim X = n$. *Then the follwoing inequality holds;*

$$h^0(K_X + (n-1)L) \geq g(L) - q(X).$$

For the time being, this conjecture is true if (X, L) is one of the following;

(1) $\dim \mathrm{Bs}\,|L| \leq 0$,
(2) $\dim X = 2$.

If Conjecture 1 is true, then the following natural problem arises;

Problem 1. *For small non-negative integer m, give a classification of quasi-polarized manifolds (X, L) with $m = g(L) - q(X)$.*

If $m=0$ and $n=2$, then this problem relates to the problem of blowing up of polarized surfaces. Let S be a smooth projective surface and let L be an ample line bundle on X. Let $p_1, \ldots, p_r$ be points on S in a general position, and let $\pi : \widetilde{S} \to S$ be blowing ups at $p_1, \ldots, p_r$. Let $a_1 \ldots, a_r$ be positive integers and $\widetilde{L} := \pi^*L - \sum_j a_j E_j$, where $E_j := \pi^{-1}(p_j)$. Then it is difficult to check that $\widetilde{L}$ is ample. For the case where $a_1 = \cdots = a_r = 1$, Yokoyama proved the following theorem;

Theorem 2. (Yokoyama) *Assume that $a_1 = \cdots = a_r = 1$ and $|L|$ has an irreducible reduced curve. If $g(L) > q(X)$, then $\widetilde{L}$ is ample.*

When we use this theorem, it is important to know the classification of polarized surfaces (S, L) with $g(L) = q(S)$.

Remark. If $\kappa(X) \geq 0$ and an irreducible reduced curve $C \in |L|$ has a singularity, then $\widetilde{L}$ is ample because $g(L) > q(S)$ in this case (see [Fk1] and [Fk2]).

Here we consider the classification of quasi-polarized manifolds (X, L) with small value $m = g(L) - q(X)$.

First we study the case in which X is a surface. Then the following facts are known;

(2-0-1) A classification of quasi-polarized surfaces (X, L) with $\kappa(X) \leq 1$ and $g(L) = q(X)$ ([Fk2]).

(2-0-2) A classification of quasi-poalrized surfaces (X, L) with $\kappa(X) = 2$, $h^0(L) \geq 1$, and $g(L) = q(X)$ ([Fk1], [Fk8]).

(2-1) A classification of polarized surfaces (X, L) with $\kappa(X) \geq 0$, $h^0(L) \geq 1$, and $g(L) = q(X) + 1$ ([Fk5]).

Next we consider the case in which $\dim X = 3$.

(3-0) A classification of polarized 3-folds (X, L) with $h^0(L) \geq 3$ and $g(L) = q(X)$ ([Fk7]). In this case (X, L) is one of the following two types;

(3-0-1) Polarized 3-folds (X, L) with $\Delta(L) = 0$. (This was classified by Fujita. See [Fj4].)

(3-0-2) A scroll over a smooth curve.

(3-1) A classification of polarized 3-folds (X, L) with $h^0(L) \geq 4$ and $g(L) = q(X) + 1$ ([Fk4]). Then (X, L) a Del Pezzo 3-fold.

By considering (3-0) and (3-1), in [Fk4] and [Fk7] the author proposed the following conjecture;

Conjecture 3. ([Fk4], [Fk7].) *Let (X, L) be a polarized manifold with $n = \dim X \geq 4$.*

(n-0) *Assume that $g(L) = q(X)$ and $h^0(L) \geq n$. Then (X, L) is a polarized manifold with $\Delta(L) = 0$ or a scroll over a smooth curve.*

(n-1) *Assume that $g(L) = q(X) + 1$ and $h^0(L) \geq n + 1$. Then (X, L) is a Del Pezzo manifold.*

By considering (3-0) and (3-1) above, we expect that we can classify polarized 3-folds (X, L) with $g(L) = q(X) + 2$ and $h^0(L) \geq 5$. The following result is one of the main theorems of the author's talk.

Main Theorem 1. *Let (X, L) be a polarized 3-fold. Assume that $h^0(L) \geq 5$ and $g(L) = q(X) + 2$. Then (X, L) is one of the following;*

(1) *A hyperquadric fibration over $\mathbb{P}^1$.*
(2) *A scroll over a smooth surface S with $q(S) = 0$.*

Remark. In each cases, the irregularity of X is zero. Hence we get $g(L) = 2$. Therefore we obtain an explicit classification of (X, L). (See [Fj2].)

Proof of Main Theorem 1. Here we get a sketch of proof of the Main Theorem 1. First assume that $K_X + 2L$ is not nef. Then (X, L) is one of the following types:

(1) $(\mathbb{P}^3, \mathcal{O}(1))$,
(2) $(\mathbb{Q}^3, \mathcal{O}(1))$,
(3) scroll over a smooth curve.

But in these cases, we obtain $g(L) = q(X)$ and this is a contradiction by hypothesis.

So we may assume that $K_X + 2L$ is nef. Let (X', L') be the first reduction of (X, L). (Let X be a smooth projective variety with $\dim X = n$ and let L be an ample line bundle L on X. Then we call that (X', L') is the first reduction of (X, L) if there exist a smooth projective variety X', an ample line bundle L' on X', and a birational morphism $\pi : X \to X'$ such that π is a blowing up at a finite set on X', $K_X + (n-1)L = \pi^*(K_{X'} + (n-1)L')$, and $K_{X'} + (n-1)L'$ is ample.)
We remark that $L^n \leq (L')^n$ in this case.
Here we use the following Theorem, which is very important for the proof of Main Theorem.

Theorem A. *Let (X, L) be a polarized 3-fold with $g(L) = q(X) + m$, $h^0(L) \geq m + 3$, and $q(X) \geq m - 1$, where m is a non-negative integer. Assume that $K_X + L$ is nef. Then $L^3 \leq 2m$.*

By using Theorem A and the theory of Δ-genus, we can prove the following Claim.

Claim B. *$K_{X'} + L'$ is not nef.*

Proof of Claim B. Assume that $K_{X'} + L'$ is nef.
If $q(X) \geq 1$, then by Theorem A, we get that $L^3 \leq (L')^3 \leq 4$.
If $q(X) = 0$, then $L^3 \leq (L')^3 \leq 2$ since $K_{X'} + L'$ is nef.
We set $t = 4 - L^3$. Then $t = 0, 1, 2$ or 3. Since $h^0(L) \geq 5$, we get

$$\begin{aligned}\Delta(L) &= 3 + L^3 - h^0(L) \\ &= 7 - t - h^0(L) \\ &\leq 2 - t.\end{aligned}$$

If $t > 0$, then $\Delta(L) \leq 1$. By using the theory of Δ-genus, we can easily get a contradiction.
So we assume $t = 0$. If $h^0(L) \geq 6$, then we get $\Delta(L) \leq 1$ and by using the same method as above, we get a contradiction.
If $h^0(L) = 5$, then $\Delta(L) \leq 2$. Here we also use the Δ-genus theory, we also get a contradiction.

Therefore $K_{X'} + L'$ is not nef. By adjunction theory, polarized manifolds (X, L) such that $K_{X'} + L'$ is not nef is classified.

(1) $K_X \sim -2L$, that is, (X, L) is a Del Pezzo manifold.
(2) A hyperquadric fibration over a smooth curve.
(3) A scroll over a smooth surface.
(4) Let (X', L') be the first reduction of (X, L).
(4-1) $(X', L') = (\mathbb{Q}^3, \mathcal{O}(2))$,
(4-2) $(X', L') = (\mathbb{P}^3, \mathcal{O}(3))$,
(4-3) X' is a $\mathbb{P}^2$-bundle over a smooth curve C with $(F', L'|_{F'}) = (\mathbb{P}^2, \mathcal{O}(2))$ for any fiber F' of it.

In the end we check these cases in detail, and we obtain the result.

Next we consider the case where $\dim X \geq 3$. In particular, we mainly consider the case in which $\mathrm{Bs}\,|L| = \emptyset$. Then we get the following results; Let (X, L) be a polarized manifold such that $\mathrm{Bs}\,|L| = \emptyset$.

(f-0) If $g(L) = q(X)$, then $\Delta(L) = 0$ or (X, L) is a scroll over a smooth curve.
(f-1) If $g(L) = q(X) + 1$, then (X, L) is a Del Pezzo manifold.

By using the method of Main Theorem 1, we get a classification of polarized manifolds (X, L) with $n = \dim X \geq 3$, $\mathrm{Bs}\,|L| = \emptyset$, and $g(L) = q(X) + 2$.

Main Theorem 2. ([Fk9]) *Let (X, L) be a polarized manifold with $\dim X = n \geq 3$. Assume that $\mathrm{Bs}\,|L| = \emptyset$ and $g(L) = q(X)+2$. Then (X, L) is one of the following type:*

(1) *X is a double covering of $\mathbb{P}^n$ with branch locus being a smooth hypersurface of degree 6, and L is the pull back of $\mathcal{O}_{\mathbb{P}^n}(1)$,*

(2) *(X, L) is a scroll over a smooth surface Y. Let $\mathcal{E}$ be a locally free sheaf of rank two on Y such that $(X, L) \cong (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$. Then $(Y, \mathcal{E})$ is either*

(2-1) *$Y \cong \mathbb{P}_\alpha{}^1 \times \mathbb{P}_\beta{}^1$ and $\mathcal{E} \cong [H_\alpha + 2H_\beta] \oplus [H_\alpha + H_\beta]$, where H_α (resp. H_β) is the ample generator of $\mathrm{Pic}(\mathbb{P}_\alpha)$ (resp. $\mathrm{Pic}(\mathbb{P}_\beta)$).*

(2-2) *Y is the blowing up of $\mathbb{P}^2$ at a point and $\mathcal{E} \cong [2H - E]^{\oplus 2}$, where H is the pull back of $\mathcal{O}_{\mathbb{P}^2}(1)$ and E is the exceptional divisor,*

(2-3) *$Y \cong \mathbb{P}(\mathcal{F})$, where $\mathcal{F}$ is a rank two vector bundle over an elliptic curve C with $c_1(\mathcal{F}) = 1$ and $\mathcal{E} = H(\mathcal{F}) \otimes p^*(\mathcal{G})$, where $p : Y \to C$ is the bundle projection and $\mathcal{G}$ is any rank two vector bundle on C defined by a non splitting exact sequence*

$$0 \to \mathcal{O}_C \to \mathcal{G} \to \mathcal{O}_C(x) \to 0,$$

where $x \in C$.

(3) *There is a fibration $f : X \to C$ over a smooth curve C with $g(C) \leq 1$ such that every fiber F of f is a hyperquadric in $\mathbb{P}^n$ and $L_F = \mathcal{O}(1)$. Then $\mathcal{E} := f_*(\mathcal{O}(L))$ is a locally free sheaf of rank $n+1$ on C, $X \in |2H(\mathcal{E}) + \pi^*(B)|$ on $\mathbb{P}(\mathcal{E})$ for some line bundle B on C, and $L = H(\mathcal{E})|_X$, where π is the projection $\mathbb{P}(\mathcal{E}) \to C$, and $H(\mathcal{E})$ is the tautological line bundle on $\mathbb{P}(\mathcal{E})$. We put $d = L^n$, $e = c_1(\mathcal{E})$, and $b = \deg B$.*

(3-1) *If $g(C) = 1$, then we have $n = 3$, $d = 6$, $e = 4$, $b = -2$, and $\mathcal{E}$ is ample.*

(3-2) *If* $g(C) = 0$, *then we have* $3 \leq d \leq 9$, $e = d - 3$, $b = 6 - d$, *and their lists are table* 2 *in* [FI].

In the end, we propose a problem which is induced from Main Theorem 1.

Problem 2. *Classify* n*-dimensional polarized manifolds with* $g(L) = q(X) + m$ *and* $h^0(L) \geq n + m$.

If $\mathrm{Bs}\,|L| = \emptyset$, $n \geq 3$, and $m \geq 0$, then we can get a classification of these polarized manifolds. We will report this in a future paper.

References

[BLP] Beltrametti, M. C., Lanteri, A., and Palleschi, M., *Algebraic surfaces containing an ample divisor of arithmetic genus two*, Ark. Mat. **25** (1987), 189–210.

[BS] Beltrametti, M. C. and Sommese, A. J., *The adjunction theory of complex projective varieties*, de Gruyter Expositions in Math. **16** (1995), Walter de Gruyter, Berlin, NewYork.

[Fj1] Fujita, T., *On polarized manifolds whose adjoint bundles are not semipositive*, Advanced Studies in Pure Math. **10** (1985), 167–178.

[Fj2] Fujita, T., *Classification of polarized manifolds of sectional genus two*, the Proceedings of "Algebraic Geometry and Commutative Algebra" in Honor of Masayoshi Nagata (1987), 73–98.

[Fj3] Fujita, T., *contribution to "Birational geometry of algebraic varieties. Open problems"*, The 23rd Int. Symp. of the Division of Math. of the Taniguchi Foundation, Katata, August 1988.

[Fj4] Fujita, T., *Classification Theories of Polarized Varieties*, London Math. Soc. Lecture Note Series, vol. 155, Cambridge University Press, 1990.

[Fk1] Fukuma, Y., *On polarized surfaces* (X, L) *with* $h^0(L) > 0$, $\kappa(X) = 2$, *and* $g(L) = q(X)$, Trans. Amer. Math. Soc. **348** (1996), 4185–4197.

[Fk2] Fukuma, Y., *A lower bound for the sectional genus of quasi-polarized surfaces*, Geom. Dedicata **64** (1997), 229–251.

[Fk3] Fukuma, Y., *A lower bound for sectional genus of quasi-polarized manifolds*, J. Math. Soc. Japan **49** (1997), 339–362.

[Fk4] Fukuma, Y., *On polarized 3-folds* (X, L) *with* $g(L) = q(X) + 1$ *and* $h^0(L) \geq 4$, Ark. Mat. **35** (1997), 299–311.

[Fk5] Fukuma, Y., *On polarized surfaces* (X, L) *with* $h^0(L) > 0$, $\kappa(X) \geq 0$ *and* $g(L) = q(X) + 1$, Geom. Dedicata **69** (1998), 189–206.

[Fk6] Fukuma, Y., *On the nonemptiness of the adjoint linear system of polarized manifolds*, Can. Math. Bull. **41** (1998), 267–278.

[Fk7] Fukuma, Y., *On sectional genus of quasi-polarized 3-folds*, Trans. Amer. Math. Soc. **351** (1999), 363–377.

[Fk8] Fukuma, Y., *On quasi-polarized surfaces of general type whose sectional genus is equal to the irregularity*, to appear in Geom. Dedicata.

[Fk9] Fukuma, Y., *On complex manifolds polarized by an ample line bundle of sectional genus* $q(X) + 2$, preprint (1998).

[FI] Fukuma, Y. and Ishihara, H., *Complex manifolds polarized by an ample and spanned line bundle of sectional genus three*, Arch. Math. **71** (1998), 159–168.

[I] Ionescu, P., *Generalized adjunction and applications*, Math. Proc. Cambridge Philos. Soc. **99** (1986), 457–472.